International Journal of Theoretical Physics, Vol. 33, No. 6, 1994

Random Graphs as a Model for Pregeometry

Frank Antonsen[1]

Received November 10, 1993

A toy model for pregeometry based on random graphs is proposed, and it is shown how it is related to other models in the literature. A prediction of the dimensionality of space is given and we obtain bounds on the Euler number.

1. INTRODUCTION

Gravitation is one of the major problems of modern theoretical physics. Ever since Einstein achieved a marriage of Newtonian gravity and special relativity we have possessed a beautiful theory, the general theory of relativity. This theory is based on a geometric interpretation of gravity: the curvature of space-time is caused by gravitation and *vice versa*, as described by the field equations equating the curvature (in the form of the Einstein tensor) to the energy-momentum tensor (Misner *et al.*, 1973). Unfortunately, the model is based on a coupling constant with dimension m^{-2}, so a further marriage with quantum theory is impossible: general relativity is not renormalizable. We find ourselves in the embarrassing position of having *two* excellent theories together explaining more or less everything from very small length scales (subnuclear) to extremely large ones (supergalactic), but which are incompatible. This has led to suggestions that gravity should not be quantized but remain classical. Such an approach must be considered most unaesthetic. Proposals have also been made deriving gravity from other forces, an approach going back to Sakharov, as an effective field theory holding at "large" distances. This approach is also known as pregeometry but differs from the philosophy we want to bring forward (Terazawa, 1991; Akama and Oda, 1991; Floreanini and Percaci, 1990).

[1]Niels Bohr Institute, University of Copenhagen, Copenhagen, Denmark.

0020-7748/94/0600-1189$07.00/0

If quantum field theory and general relativity are incompatible, then a possible way out would be to find extensions of one or the other (or both). An example of such a philosophy is superstring theory, in which the usual pointlike particles are replaced by one-dimensional objects, strings. Similarly, extensions of general relativity have been proposed. Einstein himself was working on such an extension when he died, although he was not motivated by the marriage of gravity and quantum theory, but the marriage of the two classical forces, gravity and electromagnetism. Very early proposals were put forward uniting these two by extending the dimensionality of space-time: Kaluza–Klein theory. Whether one likes or dislikes these attempts is a matter of taste (Gegenberg *et al.*, 1988; Jackiw, 1985).

Einsteinian gravity is based on differential geometry, and all gauge interactions can be cast into such a geometrical form by using vector bundles. It thus seems that geometry, from a classical point of view, is an extremely fruitful and deep concept. But what do we need in order to specify this setup? First of all we need (pseudo-) Riemannian geometry, i.e., a set locally diffeomorphic to open subsets of Minkowski space, so we need:

1. A set (the manifold).
2. A differentiable structure (the diffeomorphisms).
3. Minkowski geometry (the tangent space).
4. An affine structure (essential for the equivalence principle).

To this list one could add (a) a symplectic structure (necessary if we want Hamiltonian and/or Lagrangean dynamics) and (b) fiber bundles (to include the matter and Yang–Mills fields).

Clearly general relativity is rather "high level," in the sense that it does not work with primitive objects at all. Yang–Mills theory is even higher in that it requires a vector-bundle structure with a Lie algebra as fiber (Göckeler and Schücker, 1989). This suggests that a deeper theory could be arrived at by going to more primitive objects. The most primitive objects and the corresponding theories are:

1. Propositions; logic.
2. Categories and sets; abstract set theory.
3. Relations and compositions; algebra and topology.

When we note that many problems in quantum mechanics can be remedied when we discretize space-time (lattice regularization), we see that a fundamental quantum theory should be formulated in discrete terms (Lee, 1986). Discretized versions of general relativity, simplicial gravity, for instance, have been proposed over the years (e.g., most recently the matrix models), and sometimes this approach is also known as pregeometry. In general the word "pregeometry" is used in the following sense:

Pregeometry (Lit. "before geometry"). A model in which Einstein's theory of gravity, or some extension of it, is derived as a limiting case, even though geometrical notions were absent originally. Whether this is done using topological, algebraic, or set-theoretic models and/or by considering the graviton as a composite object is immaterial. Eventually one would hope to derive gravity from pure mathematical logic.

Here we have also written down the ultimate goal: a formulation in terms of pure mathematical logic. This idea goes back to Wheeler. Various approaches to quantum gravity based on cellular automata, quantum logics, or similar structures proposed within the last 20 years or so ('t Hooft, 1990; Zapatrin, 1993; Finkelstein, 1969; Brightwell and Gregory, 1991; Bombelli *et al.*, 1987), naturally fall within this category, even though the authors might not have thought of them as being "pregeometric." We will now propose a model.

2. SETTING UP THE MODEL

To set up a physical model we must ask ourselves many questions, the first one being: *What are the basic quantities?* The fundamental geometric concept must be that of a point.[2] This is also the point of view taken in almost all models; from models based on differential geometry (Gegenberg *et al.*, 1988; Jackiw, 1985), quantum norm theory (Isham, 1990; Isham *et al.*, 1990) or metric spaces (Alvarez *et al.*, 1988; Alvarez, 1988), to models based on causal sets ('t Hooft, 1990; Zapatrin, 1993; Finkelstein, 1969; Brightwell and Gregory, 1991; Bombelli *et al.*, 1987). We will take the point to be one of the basic quantities of our model. Now, in quantum theory we introduce operators to create and annihilate the quantities basic to the model at hand. Thus we will define point creation and annihilation operators $a, a^\dagger$. Furthermore, a correlation, or interaction, is needed to get some structure; the fundamental concept must be that of *linking* points. We will only allow single bonds between two points, and we will not allow a point to link to itself (this is not really a great loss of generality, since any other kind of graph can be spanned by one of the kinds we consider). Links will be taken as our second set of basic quantities, and we will define corresponding "second quantization" operators $b, b^\dagger$. Thus the fundamental object is a *graph* (Berge, 1973; Bollobás, 1985).

[2]Actually, as we will see later, we can recast the model in a form where the points have disappeared. For pedagogical reasons, however, it is useful to formulate this model this way.

A given set of points and links will be called a *universe*, and we can describe it by two quantities:

1. The number v of points. (The *order* of the graph.)
2. A *topological matrix* A describing the links.

This topological matrix will have the property that

$$A_{ij} = \begin{cases} 0 & (p_i, p_j) \text{ not linked} \\ 1 & (p_i, p_j) \text{ linked} \end{cases} \tag{1}$$

where p_i denotes the ith point (given some completely arbitrary labeling). The other rules which we will specify are:

1. When deleting a point we also automatically delete all of its links.
2. This being so, we choose always to delete the point with the lowest *degree*, i.e., number of links going out from it. When more points have the same minimal degree we choose one of them at random.
3. Avoid the "inhumane" situation of deleting the entire universe just by attempting to create a point or link that was already there or to delete one that was not. We forbid the application of a to the empty graph, b to the graph without links, and $b^\dagger$ to the simplex.

Any universe can be characterized by the two integers v, λ; however, this characterization is not unique, and to obtain uniqueness we need a label keeping track of the degenerate graphs, thus

$$|\psi\rangle = |v, \lambda, \kappa\rangle \tag{2}$$

The last parameter κ depends on an arbitrary enumeration of graphs with a given pair of values for v, λ. It is restricted by

$$0 < \kappa \le \binom{\binom{v}{2}}{\lambda} \tag{3}$$

Similarly, λ, the number of links, is restricted by

$$0 \le \lambda \le \Lambda \equiv \binom{v}{2} = \frac{1}{2} v(v-1) \tag{4}$$

The quantities we have referred to as "points" are purely abstract entities; they should *not* be thought of as zero-dimensional objects imbedded in a given (Euclidean or Minkowskian) space-time. *A point is just a member of a countable set*. Similarly, a "link" is nothing but a pair of points; it is not a line segment, arc, nor piece of a geodesic. Actually, the entire model can be expressed solely in terms of $\mathbb{N}$, the set of natural numbers; "God created the natural numbers, the rest is the work of man" as Kronecker said. Since a link is a pair of points, a graph can be represented in an abstract way by a pair $G = (V, E)$, where $V \subseteq \mathrm{N}$ is the set of points and $E \subseteq V \times V$ is the set of links. Of course E has to satisfy certain restraints.

It should be noted, however, that the notion of a point floating around in splendid isolation is somewhat metaphysical; such an object would not be observable.

Denote the space of all possible universes, graphs, by Γ. An appropriate name for this is *metaspace*. Consider a sequence $\{|\psi_n\rangle\}_{n\in I}$ from Γ, where I is some index set, $I \subseteq \mathrm{N}$. We will say that this sequence is an *evolution* if

$$\forall n \in I \exists \alpha_n \in \{a, a^\dagger, b, b^\dagger\}: \quad |\psi_{n+1}\rangle = \alpha_n |\psi_n\rangle \tag{5}$$

i.e., if each universe can be obtained from the one before it by the addition or removal of a point or a link. With the concept of evolution comes another, that of *time*, which is simply the index n (assuming $I = \{1, 2, \ldots, N\}$). Since our model is discrete, this definition of time is unique.

We can summarize the basic philosophy of the model so far as follows:

1. *Space is built up from fundamental objects (points and links).*
2. *Time is a parametrization of an evolution of spaces.*

Probably *the* most important concept in physics is energy, especially the Hamiltonian function. We can view this function from two angles: it is the time-development operator (the generator of translations in time) and it is also the energy of the system. While the first notion is dynamical, the latter is intrinsically static. A special case is the Hamiltonian for a static state. Here we write down the energy of a single state, i.e., not an evolution. Later we will consider dynamics further.

For our Hamiltonian we want, of course, a topological invariant. A canonical invariant for a 2-dimensional surface is the *Euler number* χ. For a graph we can also use this definition. Hence we write

$$H = H_0 + H_1 + \sum_{p \geq 3} H_p^{(2)} + \sum_{p,q} H_{p,q}^{(3)} + \cdots \tag{6}$$

where H_0 denotes the energy of the points, H_1 that of the links, and $H_p^{(2)}$ has to do with the number of 2-dimensional objects, i.e., polygons, with p sides; similarly, $H_{pq}^{(3)}$ deals with the number of polytopes built up from p- and q-gons. We assume the terms to be essentially just the number of objects with the indicated dimensionality, i.e., $H_0 \propto v$, etc.

Since the topological matrix has the property that $A_{ij} = 1$ if and only if the points i, j are linked, it follows that $(A^2)_{ij} = \sum_k A_{ik} A_{kj}$ must equal the number of paths of length two going between points i, j. In general

$$(A^k)_{ij} = |\{\text{paths between points } i, j \text{ of length } k\}| \tag{7}$$

In other words, the ith diagonal element of A^k gives the number of k-gons containing the point i. Hence we put

$$H_p^{(2)} = \frac{\alpha_p}{p!} \operatorname{Tr} A^p \tag{8}$$

where α_p is some constant. For $p \geq 3$ this gives us the number of p-gons. The trace of A^2 also has a simple interpretation. A closed path of length two must consist in going along one link and then return along the same link. Clearly $(A^2)_{ii} = d_i$, where d_i denotes the degree of the point, i.e., the number of links going out from it. Hence

$$\operatorname{Tr} A^2 = \sum_{i=1}^{v} d_i = 2\lambda \tag{9}$$

since every link connects exactly two points. We thus see that also H_1 can be written as the trace of some power of A, namely $H_1 = \alpha_1 \operatorname{Tr} A^2$. For simplicity we restrict our attention to d-simplices. For the two-dimensional case the Hamiltonian then is

$$H = \alpha_0 v + \alpha_1 \frac{1}{2} \operatorname{Tr} A^2 + \alpha_2 \frac{1}{3!} \operatorname{Tr} A^3 \tag{10}$$

This already has the form familiar from the study of matrix models (Bessis *et al.*, 1985; Bilal, 1990). Let me just restrict myself to mentioning that by introducing length and orientation of the links we get a matrix model proper; I refer to Antonsen (1992*a*), in which a comparison with the quantum metric spaces introduced by Alvarez (1988) is also made. Higher dimensions can be included by noting that three points (labeled i, j, k) lie on a triangle if and only if

$$B_{ijk} \equiv A_{ij} A_{jk} A_{ik} = 1 \qquad \text{(no summation)} \tag{11}$$

The number of simplices is then given by

$$\frac{1}{4!} \sum_{ijkl} B_{ijk} B_{jkl} B_{ijl} B_{ikl} \equiv \frac{1}{4!} \operatorname{Tr}' B^4 \tag{12}$$

In general the number of d-simplices is given by the 'trace' of the $(d+1)$th power of a d-tensor, where the trace is understood as a sum over indices so as to be equivalent to a sum over common $(d-1)$-simplices (Antonsen, 1992*a*; Froggat and Nielsen, 1991; Nielsen, 1984). Hence, all in all we have

$$H = \alpha_0 v + \frac{\alpha_1}{2}\,\mathrm{Tr}(A_{ij})^2 + \frac{\alpha_2}{3!}\,\mathrm{Tr}(A_{ij})^3 + \frac{\alpha_3}{4!}\,\mathrm{Tr}(B_{ijk})^4 + \cdots$$
$$= \sum_{d=0}^{\infty} \frac{\alpha_d}{d!}\,\mathrm{Tr}\, T_d^d \tag{13}$$

where T_d denotes the appropriate tensor ($T_0 = \delta_{ij}$, $T_1 = T_2 = A_{ij}$, $T_3 = B_{ijk}$, etc.). A similar generalization of the matrix models to $d = 3$ has been proposed by Ambjørn *et al.* (1991). Let me emphasize that the similarity with the matrix models does not serve as a justification of the model—it merely puts the former into a larger framework. Dynamics could be included by adding extra terms in the action coming from the possibility of deleting points and links, thereby changing the topology and even the dimensionality. These extra terms could be written symbolically as $A(\beta_1 a + \beta_2 a^\dagger + \beta_3 b + \beta_4 b^\dagger)A'$, where A, A' are different topological matrices. Such terms are of course absent in the much simpler matrix models.

Pregeometric models based on simplicial complexes has been studied by Lehto *et al.* (1986, 1987, 1989). See Antonsen (1992*b*) for further comments on their connection with our own pregeometric model. Once again I would like to stress that the dimension in our model is a dynamical concept: there is no fixed "background dimension," and in fact the very value of the dimensionality will necessarily undergo changes as the topological structure of the graph is changed by the operators a, $a^\dagger$, b, $b^\dagger$. The next section will show the result of some simulations.

3. SOME RESULTS

The fundamental parameters of the model are three probabilities, namely

1. The probability p_1 of creating a point.
2. The probability p_2 of deleting a point.
3. The probability p_3 of creating a link.

The probability p_4 of deleting a link is then just $p_4 \equiv 1 - p_1 - p_2 - p_3$. The task is to input values for these parameters and then at each timestep choose one operation at random with the given probability. For each of the resulting states we then calculate the number of points v, links λ and d-simplices σ_d for $d = 2, 3, 4$ (actually $v = \sigma_0$, $\lambda = \sigma_1$, so we find σ_d for $d \leq 4$). The restriction to simplices and not arbitrary cells (cubes, or, in general, polytopes) is due to limitation of resources (basically CPU time limits); it should not, however, be of great importance.

Using statistics, we are now going to derive formulas for the distribution of simplices of various dimensions in a given random graph; this can be used to derive a formula for the expectation value of the Euler number in terms of v, λ as well as giving some idea about the dimensionality. Let $\langle \sigma_p \rangle$ be the expected number of p-simplices; then, per definition, the expected value of the Euler–Poincaré characteristic is

$$\langle \chi \rangle = \sum_{p=0}^{\infty} (-1)^p \langle \sigma_p \rangle \tag{14}$$

Now, clearly, $\langle \sigma_0 \rangle = v$ and $\langle \sigma_1 \rangle = \lambda$, so the result for the two lowest dimensions is known; we now want to find the expectation values for higher dimensionality. A d-simplex consists of $d+1$ points and $\frac{1}{2}d(d+1)$ links; thus the probability is

$$\Pr(d\text{-simplex}) = \left(\frac{\lambda}{\Lambda}\right)^{d(d+1)/2} \tag{15}$$

On purely statistical grounds, then, the expected number of d-cells μ_d is given by

$$\mu_d = \Pr(d\text{-simplex}) \binom{\langle v \rangle}{d+1} = \left(\frac{\langle \lambda \rangle}{\langle \Lambda \rangle}\right)^{d(d+1)/2} \binom{\langle v \rangle}{d+1} \tag{16}$$

where, of course, $\langle \Lambda \rangle = \frac{1}{2}\langle v \rangle(\langle v \rangle - 1)$. The task is then to compute this number for some values of d ($d \leq 4$) and to compare the result with the actual number of d-simplices found in the simulated universes. Table I shows this. Only $d = 2, 3, 4$ are considered. Table II shows the same information, but this time we only average over the events in which we actually had nonzero values for v, λ. Taken together the two tables give us an idea of the dimensionality of such a random graph, and how our model differs from a pure random graph theory. We note that while for some values of the parameters there seems to be good agreement between the "theoretical" estimates μ_d and the "empirical" ones $\langle \sigma_d \rangle$, for most of the range μ_d is considerably lower than $\langle \sigma_d \rangle$ by a factor of 2 or 3 for $d = 2$, whereas for $d > 2$ the agreement is in general much worse; the expected value is often an order of magnitude below the one obtained numerically. But also notice that μ_d was a rather simple-minded statistical guess. I take this disagreement as an indication of a bias inherent in our model, probably due to the rule of always deleting the point with the lowest

Table I. The Resulting Dimension upon Application of Operators[a]

Operator	a	$a^\dagger$	b	$b^\dagger$
Effect	$d, d-1$	$d, d+1$	$d, d-1$	$d, d+1$

[a] d denotes the value of the dimensionality before the graph is operated on.

Table II. A Comparison of the Number of d-Cells from Statistical and Numerical Work

p_1	p_2	p_3	μ_2	$\langle\sigma_2\rangle$	μ_3	$\langle\sigma_3\rangle$	μ_4	$\langle\sigma_4\rangle$
1/4	1/4	1/4	0.056	0.001	0.000	0.000	0.000	0.000
		1/3	4.818	4.830	0.663	2.350	0.208	0.670
		2/5	1.870	9.520	0.131	4.795	0.002	1.400
	1/3	1/4	0.325	1.870	0.007	0.725	0.000	0.120
		1/3	1.796	5.480	0.161	1.770	0.003	0.210
		2/5	13.792	27.265	4.246	24.940	0.524	14.115
	2/5	1/4	0.054	0.120	0.000	0.000	0.000	0.000
		1/3	4.797	8.605	0.863	3.800	0.051	0.675
1/3	1/4	1/4	0.267	0.385	0.001	0.000	0.000	0.000
		1/3	2.951	9.865	0.156	5.085	0.002	1.635
		2/5	13.343	30.605	3.179	25.675	0.274	12.740
	1/3	1/4	1.872	5.910	0.100	2.205	0.001	0.310
		1/3	8.043	18.665	1.444	13.985	0.087	6.795
	2/5	1/4	3.104	8.685	0.275	3.855	0.007	0.700
2/5	1/4	1/4	1.072	1.550	0.008	0.000	0.000	0.000
		1/3	4.121	9.650	0.070	0.720	0.000	0.000
	1/3	1/4	3.909	11.425	0.101	1.515	0.000	0.000

degree; this bias tends to create a richer structure than a purely random graph would have.

The concept of dimensionality for a graph is somewhat problematic; there is no accepted canonical definition. We have in fact (at least) two concepts of dimensionality, namely $d_{\max}$, d_{eff}. The first is given by the dimension of the largest p-simplex, whereas the other deals with the distribution of cells:

$$d_{\max} = \max\{d|\langle\sigma_d\rangle \geq 0\}$$

$$d_{\text{eff}} = \max\{d|\langle\sigma_d\rangle \geq \langle v\rangle/2\}$$

Both definitions are scale-dependent; what could look like a d-dimensional space on one scale could look like a space of any other dimension on another scale. Note also the effect the basic operations a, b, $a^\dagger$, $b^\dagger$ can have on the dimensionality; if before application the graph has dimension d, then after having applied either a or b on it we can have dimension d or $d-1$, while the creation operators $a^\dagger$, $b^\dagger$ gives d or $d+1$.

Again using our simulations we can now predict the dimensionality of space-time as well as its Euler–Poincaré characteristic. In Table III we show the predicted values for the different kinds of dimensionalities. We denote by $\delta_{\max}$, δ_{eff} the estimates based on μ_d. The choice in the definition of d_{eff} is, of course, arbitrary, but it is, I believe, nonetheless reasonable; it

Table III. A Comparison of the Number of d-Cells from Statistical and Numerical Work[a]

p_1	p_2	p_3	μ_2'	$\langle\sigma_2\rangle'$	μ_3'	$\langle\sigma_3\rangle'$	μ_4'	$\langle\sigma_4\rangle'$
1/4	1/4	1/4	0.097	1.000	0.000	0.000	0.000	0.000
		1/3	5.112	7.263	0.749	6.620	0.035	2.978
		2/5	2.121	10.519	0.169	5.675	0.003	3.373
	1/3	1/4	0.502	6.032	0.016	3.816	0.000	1.000
		1/3	1.908	6.123	0.182	2.392	0.004	1.024
		2/5	14.644	29.962	4.787	28.022	0.640	15.949
	2/5	1/4	0.101	1.091	0.000	0.000	0.000	0.000
		1/3	5.097	9.456	0.947	4.270	0.063	1.063
1/3	1/4	1/4	0.399	1.540	0.002	0.000	0.000	0.000
		1/3	3.184	11.210	0.181	7.423	0.002	4.360
		2/5	14.397	33.266	3.701	27.848	0.352	18.071
	1/3	1/4	2.083	6.832	0.124	2.609	0.002	1.000
		1/3	8.416	19.342	1.581	14.957	0.101	9.503
	2/5	1/4	3.508	9.869	0.352	4.645	0.010	1.022
2/5	1/4	1/4	1.193	2.488	0.009	0.000	0.000	0.000
		1/3	4.586	11.488	0.087	1.000	0.000	0.000
	1/3	1/4	4.559	13.601	0.137	1.993	0.001	0.000

[a]Only averages over the nonzero values are used.

amounts to demanding that an observer, no matter where the observation is made, is predicted to find a dimensionality d with odds at least fifty–fifty. Table IV then shows the expected values of the Euler–Poincaré characteristics.

The maximal dimensionalities should be taken with a grain of salt; for one thing there might have been only a single d-simplex in the entire evolution to give $d_{\max} = d$, and also only $d \leq 4$ were counted. They just show the range inside which the dimension can be expected to vary. We have

$$\langle d_{\text{eff}} \rangle = \frac{37}{17} \approx 2.1$$

$$\langle d'_{\text{eff}} \rangle = \frac{42}{17} \approx 2.5$$

$$\langle \delta_{\text{eff}} \rangle = \frac{22}{17} \approx 1.2$$

$$\langle \delta'_{\text{eff}} \rangle = \frac{26}{17} \approx 1.5$$

but we also see that $d'_{\text{eff}} = 3$ has the highest probability (though it is almost

Table IV. The Estimated Values of the Dimensionalities

p_1	p_2	p_3	d_{max}	d'_{max}	δ_{max}	δ'_{max}	d_{eff}	d'_{eff}	δ_{eff}	δ'_{eff}
1/4	1/4	1/4	2	2	2	2	0	0	0	1
		1/3	4	4	4	4	2	3	2	2
		2/5	4	4	4	4	3	4	1	2
	1/3	1/4	4	4	3	3	1	3	1	1
		1/3	4	4	4	4	2	2	1	2
		2/5	4	4	4	4	4	4	3	3
	2/5	1/4	2	2	2	2	1	1	1	1
		1/3	4	4	4	4	3	3	2	2
1/3	1/4	1/4	2	2	3	2	1	1	1	1
		1/3	4	4	4	4	3	3	1	1
		2/5	4	4	4	4	4	4	2	2
	1/3	1/4	4	4	4	4	2	2	1	1
		1/3	4	4	4	4	4	4	2	2
	2/5	1/4	4	4	4	4	3	3	1	2
2/5	1/4	1/4	2	2	3	3	1	1	1	1
		1/3	3	3	3	3	1	2	1	1
	1/3	1/4	3	3	3	3	2	2	1	1

equiprobable with $d'_{eff} = 2$ or 4). Remembering that $d'_{eff} = 4$ really means $d'_{eff} \geq 4$, we see that a prediction of the effective dimensionality of the universe (i.e., of the spatial part of space-time) of a value close to 3 is reasonable.

From the number of simplices of various dimensionality we can also construct the Euler–Poincaré characteristic. Let χ_μ denote the estimate based on μ_d, i.e.,

$$\chi_\mu = \sum_{d=0}^{\infty} (-1)^d \mu_d$$

and let $\langle \chi \rangle$ denote the average found in the actual simulations. We denote the quantities calculated from nonzero values only by primes as usual. The result is shown in Table V. Here the discrepancy between the naive statistical estimate and the numerical simulation is even more pronounced; most of the estimated values lie close to zero or -1, whereas most of the numerical ones are positive and somewhere in the range 2–5. Again we see that our model has a bias toward more structure than a purely random graph.

3.1. Asymptotic Behavior

It is clearly also of interest to find the asymptotic behavior of the (effective) dimension and the other global topological characteristic, χ. We

Table V. The Expected Values of the Euler–Poincaré Characteristic

p_1	p_2	p_3	$\langle\chi\rangle$	$\langle\chi\rangle'$	χ_μ	χ'_μ
1/4	1/4	1/4	4.621	5.068	4.675	4.165
		1/3	−1.150	−0.913	−0.117	−0.130
		2/5	4.815	6.580	0.431	0.318
	1/3	1/4	2.120	3.456	1.173	0.726
		1/3	1.675	2.344	−0.607	−0.681
		2/5	5.555	6.616	−0.814	−0.776
	2/5	1/4	1.970	2.582	1.904	1.592
		1/3	1.740	2.307	0.246	0.244
1/3	1/4	1/4	5.145	5.160	5.072	4.017
		1/3	2.730	4.135	−0.878	−1.007
		2/5	4.835	9.078	−2.398	−2.363
	1/3	1/4	2.290	3.163	0.005	−0.010
		1/3	3.755	5.923	−1.034	−1.029
	2/5	1/4	1.935	2.175	−0.760	−0.895
2/5	1/4	1/4	2.140	2.539	1.655	1.235
		1/3	−0.670	−0.165	−5.549	−6.153
	1/3	1/4	2.550	3.049	−3.552	−4.134

can get the dimension as $v \to \infty$ in a simple numerical manner. We have to find that value of d for which $\mu_d \geq \frac{1}{2}v \geq \mu_{d+1}$. If we let λ grow as v^γ with $1 \leq \gamma < 2$, we get the results shown in Table VI.

We see that dimension $d = 1$ dominates; in fact, γ has to be at least 1.4 in order to get a higher dimension, but then we see that we can have very high dimensionalities in some special extreme cases. The average of the

Table VI. Asymptotic Values for the Dimension Found by Numerical Methods[a]

γ	$v = 100$	$v = 500$	$v = 1000$	$v = 5000$	$v = 10{,}000$
1.0	1	1	1	1	1
1.1	1	1	1	1	1
1.2	1	1	1	1	1
1.3	1	1	1	1	1
1.4	1	1	1	2	2
1.5	1	1	2	3	3
1.6	1	3	3	4	4
1.7	1	5	5	6	6
1.8	13	11	11	10	10
1.9	—	—	—	54	45

[a] A dash indicates that no solution was found. The numbers were found by demanding that μ_d was of the same order of magnitude as $\frac{1}{2}v$.

numbers in the last column is $74/10 = 7.4$, which is very high. Obviously this high number is due to the very last entry, namely 45; if this is ignored, we get an average of $29/9 \approx 3.2$. We see that the dimensionality tends to be about 3 (we can raise it a bit by ignoring the other extremes $\gamma = 1, 1.1$) when the graph becomes very large. But we should keep in mind that the statistical estimate μ_d has a tendency to be lower than the actual number of d-cells $\langle \sigma_d \rangle$ so we would get a dimensionality slightly higher than this value of 3, but probably not too much higher. A conservative guess would be an average dimensionality of less than 5, i.e., either 3 or 4.

A bound on the Euler–Poincaré characteristic χ_μ can also be found. Note that for $v \gg d$ we have

$$\binom{v}{d+1} \approx \frac{1}{(d+1)!}$$

so for $\lambda \sim v^\gamma$ we get

$$\begin{aligned} |\chi_\mu| &\equiv \left| \sum_{d=0}^{\infty} (-1)^d \left(\frac{2\lambda}{v(v-1)} \right)^{d(d+1)/2} \binom{v}{d+1} \right| \\ &\approx \left| \sum_{d=0}^{\infty} (-1)^d 2^{d(d+1)/2} v^{(\gamma-2)d(d+1)/2} \frac{1}{(d+1)!} \right| \\ &\leq \sum_{d=0}^{\infty} 2^{d(d+1)/2} v^{(\gamma-2)d(d+1)/2} \frac{1}{(d+1)!} \end{aligned}$$

Using Stirling's formula, we get

$$\begin{aligned} |\chi_\mu| \leq \frac{2}{(2\pi)^{1/2}} \sum_{d=0}^{\infty} \exp\left[\frac{1}{2} d(d+1) \log 2 + \frac{1}{2} (\gamma - 2) d(d+1) \log v \right. \\ \left. - \left(d + \frac{3}{2} \right) \log(d+1) - (d+1) \right] \end{aligned}$$

Keeping only the terms involving $\log v$, we obtain a sum which we can approximate by a standard integral, i.e.,

$$\begin{aligned} |\chi_\mu| &\leq \frac{2}{(2\pi)^{1/2}} \int_0^\infty \exp\left[-\frac{1}{2} (2-\gamma) \log v \, (x^2 + x) \right] dx \\ &= 2\left[\frac{1}{(2-\gamma) \log v} \right]^{1/2} \exp\left(\frac{2-\gamma}{8} \log v \right) \operatorname{erfc}\left(\left(\frac{2-\gamma}{8} \log v \right)^{1/2} \right) \end{aligned} \quad (17)$$

where $\operatorname{erfc}(x)$ is the complementary error function.

We can get lower and upper bounds on the diameter as follows. Assume that the graph has effective dimension $d_{\text{eff}} = \mathrm{d}$. A d-cube consists of

2^d points and we thus have a volume of at most $v/2^d = 2^{-d}v$. This gives a radius of at most

$$R_{\max} \sim \frac{1}{2} v^{1/d} \tag{18}$$

A lower bound can be obtained by considering the number of d-simplices μ_d. The d-volume is then simply $\frac{1}{2}\mu_d$, since each simplex has volume of one-half. So we have

$$\begin{aligned} R_{min} &\sim \left(\frac{\mu_d}{2}\right)^{1/d} \\ &= \left(\frac{\lambda}{\Lambda}\right)^{(d+1)/2} \left(\frac{v}{d+1}\right)^{1/d} \\ &\approx 2^{(d+1)/2} v^{(\gamma-2)(d+1)/2} (d+1)^{-(d+3/2)/d} \exp\left(-\frac{d+1}{d}\right) \\ &\approx 2^{(d+1)/2} v^{(\gamma-2)(d+1)/2} (d+1)^{-1} e^{-1} \end{aligned} \tag{19}$$

where we have used Stirling's formula, put $\lambda = v^{\gamma}$, and approximated $d+1$ by d, etc. For an effective dimension of three we thus get

$$\tfrac{1}{2} v^{1/3} \geq R \geq v^{2(\gamma-2)} e^{-1} \tag{20}$$

4. A MASTER EQUATION AND QUANTUM THEORY

This is a rather speculative section, but the essential idea is to find a master equation governing the evolution of the graphs. Our rules were stochastic; this is actually as it should be: the most general model would have to be indeterministic, as determinism is just a special case of indeterminism. Note also that our rules give time an arrow; since the deletion of a point implies the deletion of all its links, our processes are not directly reversible. We can always get from one graph G_1 to another G_2, but it might be easier to get from the one to the other than *vice versa*. The rules for evolution have a further implication: our system is described by a Markov chain, since we can write

$$|\psi_{n+1}\rangle = \alpha_n |\psi_n\rangle$$

where $\{|\psi_n\rangle\}$ is some evolution and where α_n is one of the four basic operators. Only the immediate past matters, so we have a Markov chain. Now, this is actually a very powerful assumption; it allows us to write down a master equation which has to be satisfied. It can be shown (Gardiner, 1985), that the transition probabilities for a Markov chain with continuous parameters satisfies the *differential Chapman–Kolmogorov equation,*

$$\frac{\partial}{\partial t}P(x,t|x',t') = -\sum_i \frac{\partial}{\partial x_i}[a_i(x)P(x,t|x',t')] + \frac{1}{2}\sum_{ij}\frac{\partial}{\partial x_i \partial x_j}[b_{ij}(x)P(x,t|x',t')] + \int [w(x|y,t)P(y,t|x',t') - w(y|x,t)P(x,t|x't')]\,dy$$

where a_i, b_{ij}, w are some functions. This equation contains the Fokker–Planck and the Liouville equations as special cases among others. We will *assume* that this equation can be taken over into our discrete setup, by substituting the derivatives with respect to x_i by variations with respect to A_{ij}. Furthermore, since the transition probability P only depends implicitly on time (via the dependence on the topological matrix A_{ij})

$$\frac{\partial}{\partial t} = \frac{\partial A_{ij}}{\partial t}\frac{\delta}{\delta A_{ij}} \equiv \alpha_{ij}\frac{\delta}{\delta A_{ij}} \tag{21}$$

The equation can then be written as

$$0 = \mathscr{H}P(A|A_0) \tag{22}$$

where

$$\mathscr{H} \equiv -\frac{\delta}{\delta A_{ij}}G_{ij;kl}\frac{\delta}{\delta A_{kl}} + V(A) + W(A) \tag{23}$$

where $G_{ij;kl}$ and V are some functions and W is an integral operator,

$$W(A)\psi(A) \equiv \int w(A|B)\psi(B)\,dB \tag{24}$$

This equation opens up a lot of possibilities, but let us for now restrict our attention to its form. When we note that the topological matrix A_{ij} is related to the metric d_{ij} of the graph in a simple way,

$$d_{ij} = \min\{k|(A^k)_{ij} \neq 0\}$$

we see that we can reexpress the above master equation as an equation with variations with respect to the metric. One could hope that this, when carried to the continuum limit, would give us some sort of Wheeler–DeWitt equation, suggesting that one could [in the spirit of random dynamics (Froggat and Nielsen, 1991; Nielsen, 1984)] "derive" full-fledged quantum gravity from a simple pregeometric model. This goes beyond the scope of this paper; more details can be found in Antonsen (1992*b*, n.d.-*b*). In these references one can also find a more detailed study of various other kinds of pregeometry and how they are related to this model. Now, any physical theory has an intrinsic logical structure (for classical physics the subsets of phase space form a Boolean algebra, whereas for quantum physics the closed subspaces form what is known as a quantum logic). Our model gives rise to "intuitionistic logic," which can be seen as a generalization of quantum logics as well as classical (i.e., Boolean) logics. This is not the right place to go into a discussion of this point; instead I refer to Antonsen (1992*b*, n.d.-*a*).

5. DISCUSSION AND CONCLUSION

We were led by simple arguments about physics in general to introduce a simple model for pregeometry. This model was based on an extremely general mathematical concept, namely that of a graph [even highly abstract concepts such as a category can be defined as a (directed) graph]. We defined dynamics and some computer simulations indicated that we should expect a spatial dimensionality of about 3. The asymptotic behavior of important quantities was studied analytically, and we wrote down a master equation governing the evolution of graphs according to the model; this equation had the same form as the Wheeler–DeWitt equation.

We noticed that matrix models could be considered as special cases of our own model, which is moreover not restricted to two dimensions. Furthermore, it can be argued that a number of other, perhaps less well-known models of quantum gravity, such as the topological ideas of Isham, Alvarez, Bombelli, and others, the logic/computing-oriented ideas of 't Hooft, Zapatrin, and Finkelstein, and simplicial gravity models, could all be incorporated as special cases of our own model.

ACKNOWLEDGMENTS

I greatly appreciate discussions with Enrique Alvarez, Enric Verdaguer, Yuri Kubyshin, and Holger Bech Nielsen, and I would like to thank these people for making it possible for me to visit Madrid and Barcelona. I am also grateful to David Finkelstein for his comments on this work.

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